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as the total distance is evidently twice as far as the distance B must cover to be abreast of A. This would seem to be an elliptic integral and hence not easily evaluated.

The same reasoning would hold on the return, except for signs, and we obtain the same expression for s except for the sign of v_1x, which would be plus.

Note.—This problem should evidently have been listed under Calculus.

432. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Having given a tetrahedron, a, b, c, d, e, f, find an expression for the radius of the sphere which is tangent to the six edges.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

First, a solution is not possible in the general case. Eight spheres can be drawn to touch any four of the six edges, but none of them will touch the other two edges unless certain conditions are satisfied.

If the tetrahedron is of such a shape that a sphere can be *inscribed* to touch all the edges, the planes of the four faces of the tetrahedron cut the sphere in circles inscribed to the triangles which form the faces. At each point where one of these four circles touches an edge, another of the circles also touches it. Now if T is the point where the in-circle of the triangle ABC, whose edges are a, b, c touches the side AC, then $CT = (a + b - c)/2$; and if T is the point where the in-circle of the triangle ACD whose edges are f, d, b touches the side AC, then $CT = (b + f - d)/2$.

Therefore $a + d = c + f$.

Similarly it can be shown that $a + d = b + e$.

Hence, a sphere can be drawn to touch the six edges internally if

$$a + d = b + e = c + f \tag{1}$$

Similarly it can be shown that a sphere can be drawn tangent to the six edges, touching b, c, d *produced,* if $a - d = e - b = f - c$; that a sphere can be drawn tangent to the six edges, touching a, b, c *produced* if $a - d = e - b = c - f$; one touching a, b, f, *produced* if $a - d = b - e = f - c$; one touching e, d, f *produced* if $a - d = b - e = c - f$.

Thus, if either

$$a + d = b + e = c + f,$$

$$\text{or}\quad a - d = b - e = c - f, \qquad \text{or}\quad a - d = b - e = f - c,$$

$$\text{or}\quad a - d = e - b = c - f, \qquad \text{or}\quad a - d = e - b = f - c,$$

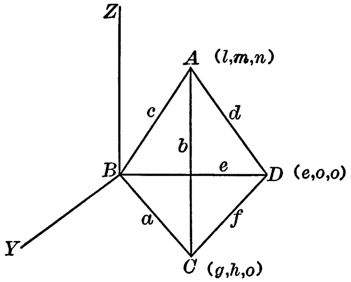
a sphere can be found to touch the six edges.

If any face is an equilateral triangle and the other three edges are equal, two such spheres can be drawn, one inscribed, the other escribed, touching the sides of the equilateral triangle and the three other edges produced.

If the opposite edges are equal in pairs (and this includes the case of the regular tetrahedron), the inscribed and four escribed spheres can be drawn.

Secondly, to find the radius of a sphere touching the edges a, c, e, f. Take BD for axis of X, BY perpendicular to BD in the plane of the triangle BCD, BZ perpendicular to BD and BY. Take C to be the point (g, h, o), A to be (l, m, n).

The centre of the sphere whose radius is required is equidistant from BD, BC, CD, AB.



Now the equations of the locus of points equidistant from BD and BC are

$$\begin{cases} hx - (a + g)y = 0, \\ hx + (a - g)y = 0, \end{cases}$$

the equations of the locus of points equidistant from CD and BD are

$$\begin{cases} hx + (e - g + f)y = eh, \\ hx + (e - g - f)y = eh, \end{cases}$$

and the equations of the locus of points equidistant from AB and BD are

$$\begin{cases} (c - l)x - my - nz = 0, \\ (c + l)x + my + nz = 0. \end{cases}$$

Taking the first of each pair of equations, we find that the center of the sphere *inscribed* to the tetrahedron touching these four edges is the point

$$\left(\frac{e(g + a)}{a + e + f}, \frac{eh}{a + e + f}, \frac{e[(c - l)(a + g) - hm]}{n(a + e + f)}\right).$$

The radius, R, of this sphere is given by the equation:

$$R = \frac{e}{n(a + e + f)} \sqrt{h^2n^2 + [(c - l)(a + g) - hm]^2}. \tag{2}$$

We must now eliminate g, h, l, m, n from (2), making use of the relations $l^2 + m^2 + n^2 = c^2$, $g^2 + h^2 = a^2$, $(e - g)^2 + h^2 = f^2$, $(l - e)^2 + m^2 + n^2 = d^2$, $(l - g)^2 + (m - h)^2 + n^2 = b^2$.

From these five equations can be obtained the following five: $g = (a^2 + e^2 - f^2)/2e$, $l = (c^2 + e^2 - d^2)/2e$, $gl + hm = (c^2 + a^2 - b^2)/2$, $h^2 = (2a^2e^2 + 2e^2f^2 + 2f^2a^2 - a^4 - e^4 - f^4)/4e^2$, $n^2 = [a^2d^2(e^2 + f^2 - a^2) + b^2e^2(f^2 + a^2 - e^2) + c^2f^2(a^2 + e^2 - f^2) - a^2(d^2 - b^2)(d^2 - c^2) + c^2(b^2 - e^2)(b^2 - d^2) - f^2(c^2 - d^2)(c^2 + b^2) - a^2e^2f^2] \div [2a^2e^2 + 2e^2f^2 + 2a^2f^2 - a^4 - e^4 - f^4]$.

Substituting these values in (2), we obtain,

$$R = \frac{1}{2n}\,\frac{1}{a + e + f}\sqrt{[(a + e)^2 - f^2][(e - c)^2 - d^2][(c - a)^2 - b^2]}.$$

Now the volume of the tetrahedron is $V = ehn/6$.

Using this fact, a little more reduction leads to the result

$$R = \frac{(a+e-f)}{24V}\sqrt{[(e-c)^2-d^2][(c-a)^2-b^2][f^2-(a-e)^2]}, \tag{3}$$

where

$$\begin{aligned}144V^2 &= a^2d^2(e^2+f^2-a^2)+b^2e^2\,(f^2+a^2-e^2)+c^2f^2(a^2+e^2-f^2)\\ &\quad -a^2(d^2-b^2)(d^2-c^2)-e^2(b^2-c^2)(b^2-d^2)-f^2(c^2-d^2)(c^2-b^2)-a^2e^2f^2.\end{aligned} \tag{4}$$

Seven similar expressions can be found for the radii of spheres tangent to the four edges a, c, e, f, touching one or more of the edges produced beyond the tetrahedron.

Thirdly, applying the conditions (1) to (3), we can obtain the required answer in a symmetrical form. Now (3) is the radius of a sphere touching the four edges a, c, e, f. If this sphere touches also the edges b, d, that is, if $a+d=b+e=c+f$, then

$$R = \frac{1}{24V}\sqrt[3]{\begin{aligned}&(a+b-c)(a-b+c)(-a+b+c)(b+d-f)\\ &(b-d+f)(-b+d+f)(c+d-e)(c-d+e)\\ &(-c+d+e)(a+e-f)(a-e+f)(-a+e+f),\end{aligned}} \tag{5}$$

where V has the value given in (4), which is already expressed in a symmetrical form. This is the required solution.

It may be added that, if A_1, A_2, A_3, A_4 are the areas of the four face-triangles, and r_1, r_2, r_3, r_4 the radii of the circles inscribed to these triangles, the above expression takes the simple form

$$R = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_1r_2r_3r_4}. \tag{6}$$

If the tetrahedron is regular the expression reduces to

$$R = \frac{a}{2\sqrt{2}}.$$

Similarly the radii of the four spheres R_1, R_2, R_3, R_4 which touch three edges and the other three edges produced, are

$$R_1 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_1r_{21}r_{31}r_{41}}; \qquad R_2 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_2r_{12}r_{32}r_{42}};$$

$$R_3 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_3r_{13}r_{23}r_{43}}; \qquad R_4 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_4r_{14}r_{24}r_{34}},$$

where r_{pq} is the radius of the circle escribed to the triangle A_p which touches the edge common to A_p and A_q and the other two edges of A_p produced.

If the tetrahedron is regular, $R_1 = R_2 = R_3 = R_4 = \dfrac{3a}{2\sqrt{2}}$.

CALCULUS.

358. Proposed by C. N. SCHMALL, New York City.

About a given circle circumscribe the smallest parabola.

SOLUTION BY HORACE OLSON, Chicago, Illinois.

We assume that by the smallest parabola is meant the smallest segment of a parabola having its bounding ordinate tangent to the given circle.